The Use of MLE in Sequential Chi-square Tests for Embedded Samples

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ABSTRACT

A Laplacian analysis of the distribution of sequential chi-square statistics for embedded samples is developed in this paper. There are found the limiting Laplace transformation of such vector statistics when the unknown parameter is replaced to MLE by the first and the last samples.

Keywords: Embedded samples; Sequential chi-square test; Vector statistics; MLE by the first sample; MLE by the last sample; Laplace transformation; Xolesky's decomposition; Limiting distributions.

1. Introduction

Zakharov, Sarmanov and Sevastyanov (1969), hereafter referred to as ZSS, suggested chi-square type sequential test based on the vector statistics

$$X_n^2 = \left(X_{n_1}^2, X_{n_2}^2, \ldots, X_{n_m}^2\right)^T, \tag{1}$$

where $n_1 < n_2 <\ldots< n_m$ - increasing sizes of embedded one to another samples, $X_{n_i}^2$ - Pearson's standard statistics. ZSS (1969) obtained the limiting Laplace transformation of statistics (1) under simple null hypothesis and convergence alternatives. Joint distribution of components of statistic X_n^2 was established Jensen (1974).

It is actually to explore the large-sample behavior of statistics

$$X_n^2(\theta,\varphi) = \left(X_{n_1}^2(\theta,\varphi), X_{n_2}^2(\theta,\varphi), \ldots, X_{n_m}^2(\theta,\varphi)\right)^T \tag{2}$$

for composite null hypothesis $H_0 : \eta = \eta_0$ against the sequence of convergence hypothesis $H_{1n} : \eta_n = \eta_0 + n^{-1/2}\gamma_n$, where $\gamma_n \to \gamma$ for fixed $\gamma \in R^p$ as $n_1 \to \infty$.

There are three practically interesting moments of this consideration. The first – in all components of statistics (2) θ replaced to θ_{n_1}, calculated by the first sample size n_1, secondly – in each components $X_{n_i}^2$ θ replaced to θ_{n_i}, calculated by sample size n_i $(i = 1,\ldots, m)$ and the third – as the first case θ replaced to θ_{n_m}, calculated by the last sample size n_m. Formanov and Mirvaliev (2000) studied case two for sequential generalized χ^2 type statistics (see also Moore and Spruill (1975)) and wide class of estimates. Mirvaliev (2002) consider the first and the third cases for multinomial MLE (or asymptotically equivalent estimates). We assume that the unknown parameter θ in (2) is replaced by MLE $\hat{\theta}_{n_1}$ and $\hat{\theta}_{n_m}$. A Laplacian analysis for this cases is provided in the next section. We conclude the article in section 3 by summarizing some remarks.

2. Laplacian Analysis

Let

$$\rho_{ij} = (n_i/n_j)^{-1/2},\ i \le j,\ \rho_i = \rho_{ii+1},\quad i, j = 1,\ldots, m \tag{3}$$

For simplicity we agreed to take, that $\rho_0 = \rho_m = \rho_{0j} = 0,\quad j = 1,\ldots, m$ and we preserve the notation ρ_{ij} for limiting values of (3). Xolesky's decomposition for the $m \times m$ - symmetric matrix R with elements (3) be C - upper triangular matrix with elements

$$c_{ii} = \sqrt{1-\rho_{i-1}^2},\ c_{ij} = \sqrt{1-\rho_{i-1}^2}\,\rho_i\rho_{i+1}\ldots\rho_{j-1},\ i < j, i = 1,\ldots, m;\ j = 2,\ldots, m.$$

Let also

$$V_m = V_m(s) = \left|I_m + 2CD(s)C^T\right|,\ \ W_{m-1} = c_m^T(I_m + 2CD(s)C^T)^{-1}c_mV_m, \tag{4}$$

where $D(s)$ is the diagonal matrix with elements $s_1, s_2, ..., s_m$ and $c_m = \left(c_m^{(1)}, ..., c_m^{(m)}\right)^T$ is m - vector with components

$$c_m^{(j)} = \rho_{jm}\sqrt{1-\rho_{j-1}^2}, j = 1,...,m.$$

2.1. MLE by the First Sample

For real λ we define

$$D_{1,\lambda} = \lambda I_m + (1-\lambda) C_0^T C_0, \tag{5}$$

where C_0 is upper triangular matrix with elements

$$_{(0)}c_{1j} = 0,\ j = 1,...,m,\ _{(0)}c_{ii} = \frac{1}{\rho_{i-1}},\ _{(0)}c_{ij} = \frac{\sqrt{1-\rho_{1i}^2}\sqrt{1-\rho_{j-1}^2}}{\rho_{1j}},\ i < j,\ i,j = 2,...,m \tag{6}$$

That is

$$\left|D_{1,\lambda}\right| = \lambda\beta\rho_{1m}^{-2} \tag{7}$$

and if $\lambda \neq 0$ and $\lambda \neq \left(1-\rho_{1m}^2\right)^{-1}$, than

$$D_{1,\lambda}^{-1} = I_m + (\lambda\beta)^{-1}(1-\lambda)^2 e_1 e_1^T + \beta^{-1}(1-\lambda)\rho_{1m}\left(e_1 c_m^T + c_m e_1^T\right) - \beta^{-1}(1-\lambda) c_m c_m^T, \tag{8}$$

where

$$\beta = 1 - \lambda + \lambda\rho_{1m}^2, \tag{9}$$

components of c_m is defined by (4) and $e_1 = (1,0,...,0)^T$ is m-vector.

Theorem 1. Suppose that $n_1 \to \infty$ such that $0 < \rho_i < 1,\ i = 1,...,m-1$. If C1, C2, C3, C4 with n replaced by n_1 and C5 (see Mirvaliev (2000)) hold, then the Laplace transformation of statistics $X_n^2(\hat{\theta}_{n_1}, \varphi_n)$

$$E\exp\left\{-\left(s_1 X_{n_1}^2(\hat{\theta}_{n_1}, \varphi_n) + s_2 X_{n_2}^2(\hat{\theta}_{n_1}, \varphi_n) + ... + s_m X_{n_m}^2(\hat{\theta}_{n_1}, \varphi_n)\right)\right\}$$

under $\left(\theta_0, \eta_{n_m}\right)$ has limiting form

$$L_{m,1}(s) = \left(V_m(s)\right)^{-\frac{r-1-t}{2}} \exp\left\{-\frac{\|\mu_1\|^2}{2}\left(1 - \frac{W_{m-1}}{V_m}\right)\right\} \bullet$$
$$\bullet \prod_{j=1}^{t}\left(V_m^{(1)}(s;\lambda_j)\right)^{-1/2} \exp\left\{-\frac{1}{2}\sum_{j=1}^{t}\vartheta_j^T\left[D_{1,\lambda_j} - \left(D_{1,\lambda_j}^{-1} + 2CD(s)C^T\right)^{-1}\right]\vartheta_j\right\}, \tag{10}$$

where

$$\mu_1 = \left[I_r - qq^T - B(B^T B)^{-1} B^T\right] B_{12}\gamma, \tag{11}$$

$$V_m^{(1)}(s;\lambda) = \left|D_{1,\lambda}\right|\left|D_{1,\lambda}^{-1} + 2CD(s)C^T\right|, \tag{12}$$

$$\vartheta_j = D_{1,\lambda_j}^{-1}\left(c_m \nu_{r-1-t+j}^{(1)} - (1-\rho_{1m}) u_m \nu_{r-1-t+j}^{(2)}\right),\ j = 1,...,t, \tag{13}$$

u_m is m - vector with components

$$u_m^{(i)} = \sqrt{1-\rho_{i-1}^2}\left(\rho_{1i}\right)^{-1}, i = 1,...,m,$$

$\nu_i^{(1)},\ \nu_i^{(2)}\ \ (i = 1,...,m)$ are components of

$$\nu^{(1)} = P^T S^T (B_{12} - BJ^{-1}J_{12})\gamma, \quad \nu^{(2)} = P^T S^T BJ^{-1}J_{12}\gamma, \tag{14}$$

$\lambda_j = \lambda_{r-1-t+j}^{(1)}, (j = 1,...,t)$, $\lambda_{r-t}^{(1)}, ..., \lambda_{r-1}^{(1)}$ are the characteristic roots of the equation

$$\left|B^T B - (1-\lambda^{(1)}) J\right| = 0, \tag{15}$$

$D_{1,\lambda}$, $\left|D_{1,\lambda}\right|$ и $D_{1,\lambda}^{-1}$ done by (5), (7) and (8).

Corollary 1. Let $m=1$. That is, we wont obtain limiting Laplace transformation of Chernoff- Lehmann statistics $X_n^2(\hat{\theta}_n,\varphi_n)$ under (θ_0,η_n) (so $n_1=n$).

In this case

$$L_{1,1}(s)=(1+2s)^{-\frac{r-1-t}{2}}\exp\left\{-\frac{\|\mu_1\|^2}{2}\left(1-\frac{1}{1+2s}\right)\right\}\bullet$$
$$\bullet\prod_{j=1}^{t}\left(V_1^{(1)}(s;\lambda_j)\right)^{-1/2}\exp\left\{-\frac{1}{2}\sum_{j=1}^{t}\frac{\left[\nu_{r-1-t+j}^{(1)}\right]}{\lambda_j}\left(1-\frac{1}{V_1^{(1)}(s;\lambda_j)}\right)\right\}, \tag{16}$$

where

$$V_1^{(1)}(s;\lambda)=1+2\lambda s.$$

Corollary 2. Let now $m=2$. Then at performance of conditions of the theorem 1 limiting Laplace transformation of statistics $\left(X_{n_1}^2(\hat{\theta}_{n_1},\varphi_n),\,X_{n_2}^2(\hat{\theta}_{n_1},\varphi_n)\right)^T$ it will be agrees to the formula (16) under $\left(\theta_0,\eta_{n_2}\right)$ has the form

$$L_{2,1}(s_1,s_2)=\left(V_2(s_1,s_2)\right)^{-\frac{r-1-t}{2}}\exp\left\{-\frac{\|\mu_1\|^2}{2}\left(1-\frac{W_1}{V_2}\right)\right\}\bullet$$
$$\bullet\prod_{j=1}^{t}\left(V_2^{(1)}(s_1,s_2;\lambda_j)\right)^{-\frac{1}{2}}\exp\left\{-\sum_{j=1}^{t}\frac{\left[s_1+s_2\frac{1}{\rho_1^2}+2s_1s_2\frac{1-\rho_1^2}{\rho_1^2}\right]\nu_j^2}{V_2^{(1)}(s_1,s_2;\lambda_j)}\right\}, \tag{17}$$

where W_1 and V_2 are defined by (4),

$$V_2^{(1)}(s_1,s_2;\lambda_j)=1+2s_1\lambda_j+2s_2\beta_{2,j}^{(1)}+4s_1s_2\lambda_j\beta_{2,j}^{(1)}(1-\rho_1^2), \tag{18}$$

$$\beta_{2,j}^{(1)}=1+\frac{1-2\rho_1^2}{\rho_1^2}\left(1-\lambda_j\right),\ j=1,\ldots,t \tag{19}$$

and

$$\nu_j=\rho_1\nu_{r-1-t+j}^{(1)}-(1-\rho_1)\nu_{r-1-t+j}^{(2)},\ j=1,\ldots,t. \tag{20}$$

2.2. MLE by the Last Sample

Theorem 2. Suppose that $n_1\to\infty$ such that $0<\rho_i<1,\ i=1,\ldots,m-1$. If C1, C2, C3, C4 with n replaced by n_m and C5 (see Mirvaliev (2000)) hold, then the Laplace transformation of statistics $X_n^2(\hat{\theta}_{n_m},\varphi_n)$

$$E\exp\left\{-\left(s_1X_{n_1}^2(\hat{\theta}_{n_m},\varphi_n)+s_2X_{n_2}^2(\hat{\theta}_{n_m},\varphi_n)+\ldots+s_mX_{n_m}^2(\hat{\theta}_{n_m},\varphi_n)\right)\right\}$$

under $\left(\theta_0,\eta_{n_m}\right)$ has limiting form

$$L_{m,3}(s)=\left(V_m(s)\right)^{-\frac{r-1-t}{2}}\exp\left\{-\frac{\|\mu_1\|^2}{2}\left(1-\frac{W_{m-1}}{V_m}\right)\right\}\bullet$$

$$\bullet\prod_{j=1}^{t}\left(\lambda_jV_m+(1-\lambda_j)W_{m-1}\right)\exp\left\{-\frac{1}{2}\sum_{j=1}^{t}\frac{\nu_j^2}{\lambda_j}\left(1-\frac{W_{m-1}}{\lambda_jV_m+(1-\lambda_j)W_{m-1}}\right)\right\}, \tag{21}$$

where V_m and W_{m-1} are given by (4), $\lambda_j = \lambda^{(1)}_{r-1-t+j}, (j=1,...,t)$, $\lambda^{(1)}_{r-t},...,\lambda^{(1)}_{r-1}$ are the characteristic roots of equation (15), $\nu_j = \nu^{(1)}_{r-1-t+j}$, $j=1,...,t$, $\nu^{(1)}_i$ are the components of $\nu^{(1)}$ (see. (14)) and μ_1 is defined by (11).

Corollary 3. Let m =2. That is

$$L_{2,3}(s_1,s_2) = \left(V_2(s_1,s_2)\right)^{-\frac{r-1-t}{2}} \exp\left\{-\frac{\|\mu_1\|^2}{2}\left(1-\frac{W_1}{V_2}\right)\right\} \bullet$$
$$\bullet \prod_{j=1}^{t} \left(V_2^{(2)}(s_1,s_2;\lambda_j)\right)^{-\frac{1}{2}} \exp\left\{-\sum_{j=1}^{t} \frac{\left[s_1\rho_1^2 + s_2 + 2s_1s_2\left(1-\rho_1^2\right)\right]\nu_j^2}{V_2^{(2)}(s_1,s_2;\lambda_j)}\right\}, \tag{22}$$

where

$$V_2^{(2)}(s_1,s_2;\lambda_j) = 1 + 2s_1\beta^{(2)}_{1,j} + 2s_2\lambda_j + 4s_1s_2\lambda_j(1-\rho_1^2), \tag{23}$$

and

$$\beta^{(2)}_{1,j} = 1 - \left(1-\lambda_j\right)\rho^2_{1m},\ j=1,...,t.. \tag{24}$$

3. Concluding Remarks

Sequential chi-square test, suggested by ZSS (1969), in the practice purpose useful, but calculate of the percentage values are difficult even $m=2$. The put $s_1 = 0$ and $s_2 = 0$ in the (17) and (22) gives that the statistics $X^2_{n_2}(\hat{\theta}_{n_1},\varphi_n)$ and $X^2_{n_1}(\hat{\theta}_{n_2},\varphi_n)$ under (θ_0,η_{n_2}) have limiting distributions

$$\chi^2_{r-1-t}\left(\|\mu_1\|^2\right) + \sum_{j=1}^{t} \beta^{(1)}_{2,j}\chi^2_{1j}\left(\frac{\nu_j^2}{\beta^{(1)}_{2,j}}\right) \text{ and } \chi^2_{r-1-t}\left(\rho_1^2\|\mu_1\|^2\right) + \sum_{j=1}^{t} \beta^{(2)}_{1,j}\chi^2_{1j}\left(\frac{\rho_1^2\nu_j^2}{\beta^{(2)}_{1,j}}\right),$$

where $\beta^{(1)}_{2,j}$ and $\beta^{(2)}_{1,j}$ are given by (19) and (24) correspondently.

It is necessary especially to note, that for $\hat{\theta}_{n_1}$ and $\hat{\theta}_{n_2}$ we have the effect of estimation by "half" observations of sample (see Mirvaliev (1987, 1990)). And at the last the power of some modified chi-square tests was investigated by Voinov and at all (2003).

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